Analytical Revision on the Proofs for Comonotone Additvity and Sub-additivity of Distorted Risk Measures

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Abstract

In financial and insurance markets no-arbitrage argument is an important condition which can be achieved by additivity property in suggested risk measures and pricing models. In this paper, I have provided some discussions about revision of previous proofs for addtitivity of dependent comonotone risks and sub-additivity property of premium principles under distortion. Four defined properties of a distortion operator in hand, I have bring a complete proof for additivity of comonotone risks in distorted risk measures which may be used as a premium principle in insurance. The key concept in the proof is that $S_{T(x)}^{-1}(p) = T(S_X^{-1}(p))$, where $T: R \to R$ is an increasing continuous function and S^{-1} is generalized inverse function of decumulative distribution function. I examined also the provided proof of sub-additivity by Wirch and Hardy, 1999 and complete the relative theorems.

Keywords: additivity, sub-additivity, distortion operator, premium principle, decumulative distribution function, correlation order, stop-loss order.

1 Introduction

By a simple definition, a risk measure is a function that allocates a non-negative real number to a risk. Many risk measures have been suggested to quantifying financial and insurance risks, but there are some important considerations to measure the insurance risks which are not the alike with the financial risk measuring. Financial pricing models cannot be applied truly for pricing insurance risks, because of some fundamental differences between these two types.

Distorted risk measured have been introduced and developed in order to find a universal framework for pricing financial and insurance risks. Considerable efforts have been made by actuaries and financial economists to build bridges to connect financial and insurance pricing theories, which have been developed after Yaari's dual theory under risk in 1987. see D'Arcy and Doherty (1988), Gerber and Shiu (1994). put together various pieces of pricing theory puzzles, an overall picture has not emerged yet.

In the insurance literature, the appropriate risk measures to pricing insurance risks are premium principles. By an (insurance) risk we mean a non-negative random variable representing the amount paid to a policyholder or his/her beneficiary, and/or a third party. We denote the set of such random variables by Υ. The premium related to an insurance risk (i.e. loss random variable) $X \in \Upsilon$ will denoted by $H(X)$.

There exist some mathematical and economical requirements such as extra loading, maximal loss, translation and scale invariance, monotonicity, first-order stochastic dominance, stop-loss ordering and continuity for premium principles which are needed to meet natural conditions of insurance contracts and ratemaking approaches while some others are just desirable.

Traditional premium principles such as Statndard deviation, Exponential, Dutch, Swiss principles and principle of Equivalent Utility couldn't preserve some of basic requirements of a desirable premium principle especially extra loading and commonly were based on the first two moments. Since loss distributions are often skewed, first two moments are not completely efficient about the level of the insurance risk. For confronting and dealing with above problems and inefficiencies in referred principles, Yaari (1987) postulated that decision makers should adjust their assumed probability distributions according to their risk-aversion for probabilities, and then make decision based on that adjusted probability distribution.

This idea leads to using a general class of "**Distortion Operators**" defined by function g with some particular properties for pricing both financial and insurance risks. Considerable efforts have been made by actuaries and financial economists to build bridges to connect financial and insurance pricing theories, which have been developed after Yaari's dual theory under risk in 1987. see D'Arcy and Doherty (1988), Gerber and Shiu (1994).

In and special view, Venter's (1991) observed that such an integration between insurance and financial pricing models may have come into reality by transforming the distribution of risk and proved it in case of insurance layers. Wang (1995, 1996) proposed calculating insurance premium by transforming the decumulative distribution function, which turned out to coincide with Yaari's economic theory of risk. So, the related premium principle have been defined by,

$$H_g[X] = -\int_{-\infty}^{0}\{1 - g[Sx(t)]\}dt + \int_{0}^{\infty} g[Sx(t)]dt$$

where for a non-negative random variable (insurance risk) it turns into

$$H_g[X] = \int_{0}^{\infty} g[Sx(t)]dt$$

The distorted risk measure in insurance should satisfy the extra loading necessity;

$$H_g[X] \leq E[X]$$

But, the vital consideration for new pricing framework under distorted risk measure was preserving the No-arbitrage argument for calculated prices in the market. No-arbitrage argument has direct relationship with additivity property of premium principle as a risk measures. In addition, the sub-additivity property is too notable in premium evaluation and coherency or proposed risk measure.

Many actuaries have tried to verify the additivity property in general for dependent and independent risks. Although this attempts was failed for dependent risks, but they could prove additivity for a large array of risk called ***Comonotone Risks***. Wang (1995) and Kaas, Dhaene, Vyncke, Goovaerts, Denuit have done many researches on this subject during 1995 to 2004 which had been laid those foundation by a proposition by Denneberg (1994). On the other hand, for the case of sub-additivity there are some proofs by Wang (1995), Wirch & Hardy (2000) for arbitrary discrete points risk random variable. Finally, Dhaene, Vanduffel, Goovaerts, kaas & Vyncke proposed an alternative theorem for proving sub-additivity of distorted risk measures by using the comonotone additivity property and stop-loss ordering.

In this paper we are going to rearrange the complete proofs for both additivity and sub-additivity or distorted risk measures specialized in premium principles with a brief examination of previous proofs.

2 Distortion Operator

1.1.1 Definition

A function $g : [0,1] \to [0,1]$ *is called a distortion function if it satisfies the following four conditions:*

1. $0 \leq g(u) \leq 1$ *and that* $g(0) = 0$ *and* $g(1) = 1$.

2. g *is increasing (so* $g'(u) \geq 0$ *when the derivative exists).*

3. g *is concave (so* $g''(u) \leq 0$ *when the second derivative exists).*

4.$g'(0) = +\infty$

The resulting premium would be risk-adjusted if the value of $\mathrm{S_Y}(\mathrm{t}) = \mathrm{g}[\mathrm{S_X}(\mathrm{t})]$ dominates $S_X(t)$ sufficiently. The transformed decumulative function $S_Y(t)$ consists of the extra risk premium or the risk loading that adjusts the net premium obtained from the distribution of original risk X (i.e. $S_X(t)$).

Generally the distortion operator $g : [0,1] \to [0,1]$ transforms a probability distribution S_X to a new distribution S_Y . As we should be able to transform the original risk distribution, we should also be able to transform the layers' distributions at the same time, because in insurance applications, the insurers often have to determine premiums for layers. For insurance layer $X(a, a+h]$ the risk-adjusted premium easily can be obtained from:

$$H_g[X(a, a + h]] = \int_0^{\infty} g[S_{X(a,a+h]}(t)]dt = \int_a^{a+h} g[S_X(t)]dt$$

indicating that the layer premium can be obtained by applying distortion operator g either to the distribution of the layer or to the distribution of the original risk X (ground-up loss).

These assumed properties are necessary to construct a desirable premium principle. Based on some principles of insurance pricing, premium principles should meet some requirements: For example, it is desirable that for projected high loss values a premium principle should carry more risk loading than low loss values. The following observations come along with the above conditions the distortion function has:

Note I. It is known that a concave function $f:(a,b)\to\mathbb{R}$ is necessarily continuous. Thus, function g is continuous on $(0,1)$.

Note II. As g being increasing and $S_X(x)$ being decreasing, $g[S_X(x)]$ is a decreasing function.

Note III.

$$\lim_{x\to+\infty} g[S_X(x)] = g[\lim_{x\to+\infty} S_X(x)] = g(0) = 0$$

$$\lim_{x\to-\infty} g[S_X(x)] = g[\lim_{x\to-\infty} S_X(x)] = g(1) = 1$$

Note that $g[S_X(x)]$ is not necessarily a decumulative function unless we have right-continuity condition of g.

Note IV. g being continuous (Note I) and $S_X(x)$ being right-continuous, it implies that $g[S_X(x)]$ is right continuous on $(0,1)$.

Note V. It is notable that $g[S_X(x)]$ is right continuous everywhere if we assume g being continuous at 1.To see this, suppose $x\to x_0^+$; we must show that $\lim_{x\to x_0^+} g[S_X(x)] = g[S_X(x_0)]$. For this, distinguish three cases:

Case I. $S_X(x_0)] = 0$. Then, for all $x > x_0$, we have $S_X(x)] = 0$ as S_X is a decreasing function. Then $g[S_X(x)] = g[S_X(x_0)]$ for $(x > x_0)$ giving $\lim_{x\to x_0^+} g[S_X(x)] = g[S_X(x_0)]$.

Case II. $0 < S_X(x_0) < 1$. then from the right continuity of S_X, we have $0 < S_X(x) < 1$ for x's on the right side of x_0 and close to x_0. Further, we have $\lim_{x\to x_0^+} S_X(x) = S_X(x_0)$. Now the continuity of g on $(0,1)$ gives $\lim_{x\to x_0^+} g[S_X(x)] = g[S_X(x_0)]$.

Case III. $S_X(x_0)] = 1$. Then, from the right continuity of S_X and the continuity of g at 1, one gets $\lim_{x\to x_0^+} g[S_X(x)] = g[S_X(x_0)]$, finishing the claim in all cases.

3 Additivity of Comonotone Risks

Although $\mathrm{H_g}[\mathrm{X}]$ is not additive in general and may not meet the no-arbitrage argument in pricing framework under $\mathrm{H_g}[\mathrm{X}]$, but it preserves additivity for comonotone risks. In other word, distorted premium principle is additive for dependent comonotone risks. Consequently, for insurance layers, the resulted premium will be additive.

1.1.2 Definition (Comonotone Risk)

Two random variables X1 and X2 are comonotone risks if there exist a risk Z and weakly increasing functions f and g such that $X_1 = f(Z)$ and $X_2 = g(Z)$.

Comonotone risks cannot be hedged against each other. In the no-hedged condition, insurers are not willing to give a reduction in the risk load for a combined policy of two comonotone risks X_1 and X_2, thus $H(X_1 + X_2) \geq H(X_1) + H(X_2)$. On the other hand the maximum premium that insurers can charge for such a combined policy is the sum of two individual risk premiums, because otherwise the policy-holder can buy separate covers cheaper; thus we must have $H(X_1 + X_2) \geq H(X_1) + H(X_2)$. These two statements imply additivity for the comonotone risks:

$$H(X_1 + X_2) = H(X_1) + H(X_2)$$

1.1.3 Definition

Let, $X : \Omega \to \overline{\mathbb{R}}$ be a function on the set Ω. By the upper set system of X we mean the collection

$$\mathcal{M} = \{\{X > x\} \big| \, x \in \overline{\mathbb{R}}\} \cup \{\{X \geq x\} | x \in \overline{\mathbb{R}}\}$$

An analogous to definition 1.1.2 is as below

1.1.4 Definition

A class of $\mathcal{L}$ of functions $\Omega \to \overline{\mathbb{R}}$ is called comonotone if the union $\cup_{X \in \mathcal{L}} \mathcal{M}_X$ is a chain.

Clearly, a class of functions is comonotone, if and only if, each pair of functions in $\mathcal{L}$ is comonotone. The following proposition gives equivalent conditions for a pair of functions to be comonotone.

1.1.5 Proposition

For two functions $X, Y : \Omega \to \overline{\mathbb{R}}$ the following conditions are equivalent:

(i) $\{X, Y\}$ *is comonotone.*

(ii) *There is no pair $\omega_1, \omega_2 \in \Omega$ such that*

$$\begin{cases} X(\omega_1) < X(\omega_2) \\ Y(\omega_1) < Y(\omega_2) \end{cases}$$

(iii) *The set $\{(X(\omega), Y(\omega)) | \omega \in \Omega\} \subset \overline{\mathbb{R}}^2$ is a chain with respect to the usual relation in $\overline{\mathbb{R}}^2$.*

(iv) *There exist a function $Z : \Omega \to \overline{\mathbb{R}}$ and increasing functions u and v on $\overline{\mathbb{R}}$ such that*

$$X = u(Z) \quad , \quad Y = v(Z)$$

(v) *There exist continuous and increasing functions u and v on $\overline{\mathbb{R}}$ such that $u(Z) + v(Z) = Z$, $(Z \in \overline{\mathbb{R}})$ and*

$$X = u(X + Y) \quad , \quad Y = v(X + Y)$$

Proof:

(i) ⇒ (ii)

Suppose **(ii)** is not true. Then there are $\omega_1, \omega_2 \in \Omega$ such that $X(\omega_1) < X(\omega_2)$ and $Y(\omega_1) < Y(\omega_2)$. Set $A = \{X > X(\omega_1)\}$ and $B = \{Y > Y(\omega_2)\}$. Then, $\omega_2 \in A \backslash B$ and $\omega_1 \in B \backslash A$, while A and B are in $\mathcal{M}_X \cup \mathcal{M}_Y$. This shows that $\mathcal{M}_X \cup \mathcal{M}_Y$ is not a chain rejecting **(i)**.

(iii) ⇒ (i)

Assume on the contrary **(i)** is not true. Then there are $A \subset \mathcal{M}_X$ and $B \subset \mathcal{M}_Y$ satisfying $A \nsubseteq B$ and $B \nsubseteq A$. Choose , $\omega_1 \in A \backslash B$ and $\omega_2 \in B \backslash A$. Either $A = \{X | X > a\}$ or $A = \{X | X \geq a$. In the first case, $\omega_1 \in A$ and $\omega_2 \notin A$ we get $\begin{cases} X(\omega_1) > a, \\ X(\omega_2) \leq a \end{cases}$ giving us the inequality $X(\omega_1) > X(\omega_2)$. In the second case, from $\omega_1 \in A$ and $\omega_2 \notin A$ we have $\begin{cases} X(\omega_1) \geq a, \\ X(\omega_2) < a \end{cases}$ giving us the inequality

$X(\omega_1) > X(\omega_2)$. In either cases, we have $X(\omega_1) > X(\omega_2)$. Similarly, we have $Y(\omega_1) > Y(\omega_2)$. Those two reject **(ii)**.

(ii)* ⇒ *(v)

Assume there are ω_1 and ω_2 such that

$$X(\omega_1) + Y(\omega_1) = X(\omega_2) + Y(\omega_2)$$
$$X(\omega_1) - X(\omega_2) = Y(\omega_1) - Y(\omega_2)$$

If we had $X(\omega_1) < X(\omega_2)$, then we would have $Y(\omega_2) < Y(\omega_1)$ contradicting **(ii)**. So, $X(\omega_1) \geq X(\omega_2)$. Similarly, the case $X(\omega_1) > X(\omega_2)$ ends up in a contradiction. Therefore,

$$X(\omega_1) = X(\omega_2)$$

and

$$Y(\omega_1) = Y(\omega_2)$$

Now set $Z = X + Y$. Then the functions $\begin{cases} u: Z(\Omega) \longrightarrow \mathbb{R}, \\ v: Z(\Omega) \longrightarrow \mathbb{R} \end{cases}$ defined by $\begin{cases} u(Z(\omega)) = X(\omega), \\ v(Z(\omega)) = Y(\omega) \end{cases}$ are well defined. Next we prove u and v are increasing;

Suppose $Z(\omega_1) < Z(\omega_2)$. Then

$$X(\omega_1) - X(\omega_2) < Y(\omega_2) - Y(\omega_1)$$

If the right-hand side were negative, then the left-hand side is negative too, contradicting **(ii)**. Similarly, the left-hand side cannot be positive. Thus, we must have $\begin{cases} X(\omega_1) - X(\omega_2) \leq 0, \\ Y(\omega_2) - Y(\omega_1) \geq 0 \end{cases}$ i.e.

$$\begin{cases} X(\omega_1) \leq X(\omega_2), \\ Y(\omega_1) \leq Y(\omega_2) \end{cases} \text{ so } \begin{cases} u(Z(\omega_1)) \leq u(Z(\omega_2)), \\ v(Z(\omega_1)) \leq v(Z(\omega_2)) \end{cases}.$$

This proves u and v being increasing. Next, we show that u and v are continuous on $Z(\Omega)$. For this take two elements $Z(\omega_1)$ and $Z(\omega_2)$ from $Z(\Omega)$ where $Z(\omega_1) \leq Z(\omega_2)$. As we sow above, $\begin{cases} X(\omega_1) \leq X(\omega_2), \\ Y(\omega_1) \leq Y(\omega_2) \end{cases}$ so,

$$0 \leq X(\omega_2) - X(\omega_1) = Z(\omega_2) - Z(\omega_1) - \underbrace{(Y(\omega_2) - Y(\omega_1))}$$
$$\leq Z(\omega_2) - Z(\omega_1)$$

Where under braced value is always positive. Thus,

$$0 \leq X(\omega_2) - X(\omega_1) = [Z(\omega_2) - Z(\omega_1)]$$
$$0 \leq u\big(Z(\omega_2)\big) - u\big(Z(\omega_1)\big) = [Z(\omega_2) - Z(\omega_1)]$$

This shows that if $|Z(\omega_2) - Z(\omega_1)| \to 0$ then $|u(Z(\omega_2)) - u(Z(\omega_1))| \to 0$. So, u is uniformly continuous on $Z(\Omega)$. This allows extending u to a continuous function on $\overline{Z(\Omega)}$. Now the complement $\mathbb{R}\backslash\overline{Z(\Omega)}$ is a countable union of intervals $\{(a_n, b_n)\}_{n=1}^{\infty}$. On each (a_n, b_n) define u linearly maintaining $u(a_n)$ and $u(b_n)$. Then define v on (a_n, b_n) by $v(Z) = Z - u(Z)$. Then, u and v are continuous on $\mathbb{R}$ satisfying $u(Z) + v(Z) = Z$. With this extensions, we have $X = u(X+Y)$ and $Y = v(X+Y)$ on Ω.

(v)* ⇒ *(iv)

This is obvious.

(iv)* ⇒ *(iii)

If $\omega_1, \omega_2 \in \Omega$, then either $Z(\omega_1) \leq Z(\omega_2)$ or $Z(\omega_2) \leq Z(\omega_1)$. In the first case, we have $\begin{cases} u(Z(\omega_1)) \leq u(Z(\omega_2)), \\ v(Z(\omega_1)) \leq v(Z(\omega_2)) \end{cases}$ i.e. $\begin{cases} X(\omega_1) \leq X(\omega_2), \\ Y(\omega_1) \leq Y(\omega_2) \end{cases}$, while in the second case we have the opposite direction. This proves **(iii)**. This finishes the proof of the proposition.

■

1.1.6 Definition

*For a decumulative distribution function S we define the **Generalized Inverse Function** as being*

$$S^{-1}(p) = sup\{x \mid S(x) > p\} \qquad (0 < p < 1)$$

1.1.7 Lemma

$$S^{-1}(p) \leq x \quad \Leftrightarrow \quad S(x) \leq p$$

Proof:

If $x^* < S^{-1}(p)$, then from the definition of sup, the exists some x such that $S(x) > p$ and $x^* < x$. Then $S(x^*) \geq S(x) > p$ as S is decreasing. So $S(x^*) > p$. This proves one half of the claim. For the other half, suppose $S(x) > p$. Then from the right continuity of S, we have $S(x+\varepsilon) > p$ for some $\varepsilon > 0$. So $x+\varepsilon$ is in the set $\{x \mid S(x) > p\}$. Therefore, it must be bounded by the Sup of the set, that is $x+\varepsilon \leq S^{-1}(p)$. Therefore, $x < S^{-1}(p)$. This proves the other half.

■

This lemma shows that

$$S^{-1}(p) = \inf\{x \mid S^{-1}(p) \leq x\} = \inf\{x \mid S(x) \leq p\}$$

So,

$$S^{-1}(p) = \inf\{x \mid S(x) \leq p\} = \sup\{x \mid S(x) > p\}$$

For any increasing right-continuous function T, we define the generalized inverse T^{-1} by

$$T^{-1}(q) = \inf\{x \mid T(x) \geq q\}$$

The inf is taken to be $+\infty$ should the set be empty. Parts (i) and (ii) of the following proposition can be prove in a similar manner as above. So we only prove part (iii):

1.1.8 Proposition

For an increasing right-continuous function $T : \mathbb{R} \to \mathbb{R}$,

(i)

$$T^{-1}(q) = inf\{x \mid T(x) \geq q\} = sup\ \{x \mid T(x) < q\}$$

(ii)

$$T^{-1}(q) \leq x \quad \Leftrightarrow \quad q \leq T(x)$$

(iii) T^{-1} *is increasing and left-continuous.*

Proof:

(iii)

If $p < q$, then by letting $A_p = \{x \mid T(x) \geq p\}$ and $A_q = \{x \mid T(x) \geq q\}$, we have

$$A_p \supset A_q$$

$$\inf A_p \leq \inf A_q$$

$$T^{-1}(p) \leq T^{-1}(q)$$

So that T^{-1} is an increasing function. To see left-continuity, let $p_n \to p^-$. Since $p_n < p$ and since T^{-1} in increasing, for all n we have $T^{-1}(p_n) \leq T^{-1}(p)$. Therefore,

$$\boldsymbol{\lim T^{-1}(p_n) \leq T^{-1}(p)} \qquad \textbf{(*)}$$

Now let γ be any number satisfying $\gamma > T^{-1}(p)$. Then from the definition of $\inf$, there exists some x such that

$$\begin{cases} p \leq T(x), \\ \quad x < \gamma \end{cases}$$

But from $T(x) \geq p$ we equivalently have $T^{-1}(p) \leq x$. So we can further write

$$\begin{cases} \quad \boldsymbol{p \leq T(x)}, \\ \boldsymbol{T^{-1}(p) \leq x < \gamma} \end{cases} \qquad \textbf{(**)}$$

Now we have the following series of conclusions:

$$x - \gamma + T^{-1}(p) < T^{-1}(p)$$
$$T\big(x - \gamma + T^{-1}(p)\big) < p$$

$$T\big(x - \gamma + T^{-1}(p)\big) < p_n \qquad \text{for large n}$$

$$x - \gamma + T^{-1}(p) < T^{-1}(p_n) \qquad \text{for large n}$$

So,

$$\boldsymbol{x-\gamma+T^{-1}(p)<\underline{\lim}\ T^{-1}(p_n)} \qquad \textbf{(***)}$$

Now by letting $\gamma \to T^{-1}(p)$, we have $x \to T^{-1}(p)$ in (**), therefore (***) reduces to

$$\boldsymbol{T^{-1}(p)<\underline{\lim}\ T^{-1}(p_n)} \qquad \textbf{(****)}$$

Now (*) and (****) imply $lim T^{-1}(p_n) = T^{-1}(p)$ when $p_n \to p$, finishing the proof of part (iii).

1.1.9 Lemma

If T is continuous, then T^{-1} is one-to-one (and so it is strictly increasing).

Proof:

Suppose $p < q$. We want to show that $T^{-1}(p) < T^{-1}(q)$. First of all, we know that T^{-1} is increasing: $T^{-1}(p) \leq T^{-1}(q)$. So if $T^{-1}(p) < T^{-1}(q)$ does not hold, then we must have had $T^{-1}(p) = T^{-1}(q)$. Call this common value x. Then from $T^{-1}(p) < x$, we have $p \leq T(x)$. On the other hand, from the continuity of T^{-1},

$$\begin{gathered} x = T^{-1}(p) \\ x - \frac{1}{n} < T^{-1}(p) \qquad \forall n \\ T\left(x - \frac{1}{n}\right) < p \qquad \forall n \\ T(x) \leq p \qquad \text{(continuity of T)} \end{gathered}$$

The two inequalities show that $T(x) = p$. Similarly, one gets $T(x) = q$. So, $p = q$, a contradiction.

■

1.1.10 Proposition

Suppose $T:\ \mathbb{R} \to \mathbb{R}$ is an increasing continuous function. Then

$$S_{T(X)}^{-1}(p) = T(S_X^{-1}(p))$$

Proof:

It suffices to show that for every x the relation $S_{T(X)}^{-1}(p) \leq x$ holds if and only if the relation $T(S_X^{-1}(p)) \leq x$ holds. Further, this remains true if we prove it for all x but a countable number of x's. Since T^{-1} is an increasing function, its set of discontinuities is countable. So we may attempt proving this assertion for the continuity points x of T^{-1}: Now with this assumption in mind, for every $o < p < 1$, we then have the following equivalent expressions:

$$\begin{aligned}
S_{T(X)}^{-1}(p) \leq x \quad &\Leftrightarrow \quad S_{T(X)}(x) \leq p \\
&\Leftrightarrow \quad \Pr[T(X) > x] \leq p \\
&\Leftrightarrow \quad 1 - p \leq \Pr[T(X) \leq x] \\
&\Leftrightarrow \quad 1 - p \leq \lim_{n\to\infty} \Pr\left[T\left(X - \frac{1}{n}\right) < x\right] \qquad [\text{as } T \text{ is continuous}] \\
&\Leftrightarrow \quad 1 - p \leq \lim_{n\to\infty} \Pr\left[X - \frac{1}{n} < T^{-1}(x)\right] \\
&\Leftrightarrow \quad 1 - p \leq \Pr[X \leq T^{-1}(x)] \\
&\Leftrightarrow \quad 1 - p \leq 1 - \Pr[X > T^{-1}(x)] \\
&\Leftrightarrow \quad S_X(T^{-1}(x)) \leq p \\
&\Leftrightarrow \quad S_X^{-1}(p) \leq T^{-1}(x) \\
&\Leftrightarrow \quad \forall n : \quad S_X^{-1}(p) < T^{-1}\left(x + \frac{1}{n}\right) \\
&\qquad \text{as } T^{-1} \text{ is both strictly increasing and continuous at } x \\
&\Leftrightarrow \quad \forall n : \ T(\,S_X^{-1}(p)) < \left(x + \frac{1}{n}\right) \\
&\Leftrightarrow \quad T(\,S_X^{-1}(p)) \leq x
\end{aligned}$$

■

1.1.11 Theorem

If $\{X, Y\}$ *is comonotone, then for* $Z = X + Y$, *we have*

$$S_Z^{-1}(p) = S_X^{-1}(p) + S_Y^{-1}(p).$$

Proof:

As in the above theorem, there are continuous function u and v on $\mathbb{R}$ such that $u + v$ is the identity function and that

$$\begin{cases} X = u(X+Y), \\ Y = v(X+Y) \end{cases}.$$

Then

$$\begin{aligned} S_X^{-1}(p) + S_Y^{-1}(p) &= S_{u(X+Y)}^{-1}(p) + S_{v(X+Y)}^{-1}(p) \\ &= u[S_{X+Y}^{-1}(p)] + v[S_{X+Y}^{-1}(p)] \\ &= S_{X+Y}^{-1}(p) \end{aligned}$$

■

This result easily extends to the multivariate case: if $X_1, X_2, \dots, X_{n-1}, X_n$ are mutually comonotones, then

$$S_{(X_1,X_2,\dots,X_{n-1},X_n)}^{-1}(p) = \sum_{i=1}^{n} [S_{X_i}^{-1}(p)]$$

Recall that the net expected loss $E(X)$ is equal to the area below the curve S_X(t). While vertical slicing gives, $E(X) = \int_0^\infty S_X(t)dt$. On the other hand, horizontal slicing yields that,

$$E(X) = \int_0^\infty S_X^{-1}(q)dq$$

where the reverse $\boldsymbol{S_X^{-1}(q)}$ my not be uniquely defined. However, it does not affect the integration. See figure (3.1).

Remark. One can easily verify that for an increasing function g with $g(0) = 0$ and $g(1) = 1$,

$$H[X] = \int_0^\infty g[S_X(t)]dt = \int_0^\infty S_X^{-1}(q)dg(q)$$

with the aid of horizontal slicing, we can now examine the concept of comonotoneity in terms of insurance risk premium densities.

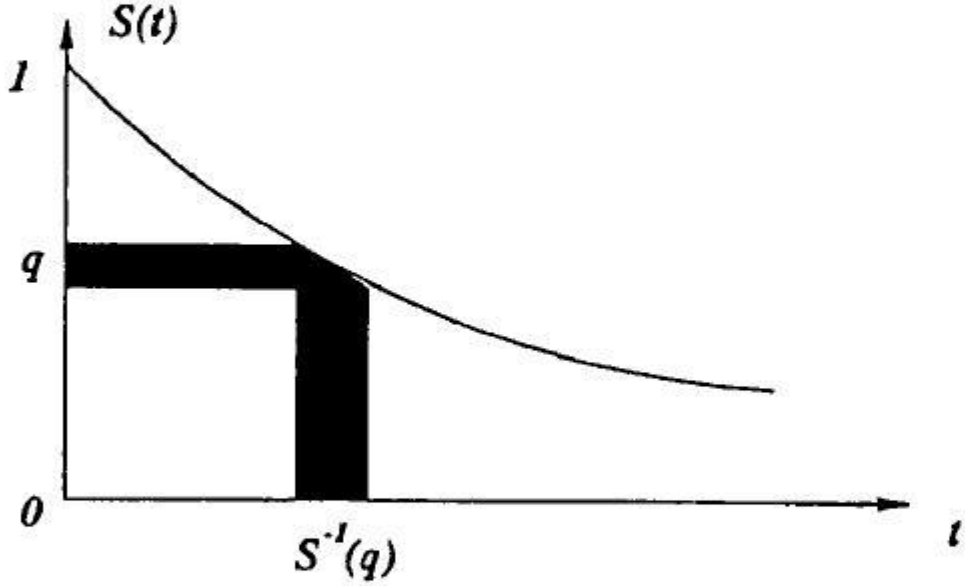

Figure 3.1: Slicing a risk vertically & horizontally

1.1.12 Theorem

The premium principle $H_g[X]$ *is additive on comonotone risks.*

Proof:

$$H(X+Y) = \int_0^\infty S_{X+Y}^{-1}(t)dg(t) = \int_0^\infty [S_X^{-1}(t) + S_Y^{-1}(t)]dg(t) = H(X) + H(Y)$$

■

4 Sub-additivity

Sub-additivity means the premium for the sum of two risks is not greater than the sum of the individual premiums and the rationale behind sub-additivity can be summarized as "a merger does not create extra risk". The property is desirable, because the buyer of insurance will not buy two merged insurance cover for two risk in a price more than two separate insurance cover from two insurer.

By general attitude joining financial literature, sub-additivity reflects the idea that risk can be reduced by diversification. When equality holds, two risks are additive. The diversification effect in its simplest form is defined as the difference between the sum of two premium (generally risk measures) of standalone risks and the premium of both risks taken together, that is,

$$H(X) + H(Y) - H(X + Y)$$

The diversification effect is always positive for subadditive risk measures. In portfolio optimization, sub-additivity and scale invariance ensures that the risk surface to be minimized in the space of portfolio is convex. Only if the surface is convex there would be a unique absolute minimum and only then will the risk minimization process always find a unique, well-diversified, optimal solution. In other word, validity of sub-additivity leaves no room for the policy-holder to split a risk into pieces.

We turn to discuss some of previous attempts to prove the sub-additivity of distorted risk measures.

4.1 Previous Discussions of sub-additivity

Wang (1995) presented a theorem for sub-additivity of a distorted risk measure where the distortion function was defined especially as $g[S_X(x)] = [S_X(x)]^{1/p}$ where $p > 1$.

1.1.13 Theorem (Wang 1995)

For any two non-negative random variables U and V without assuming independence, the following inequality holds:

$$H_g(U+V) \le H_g(U) + H_g(V)$$

The proof was based on selecting some arbitrary points U and V from a discrete domain of risk random variable $\{0,1,\ldots,n\}$ and proving the theorem by induction for these points. Then Wang have generalized the proof for all values by the approximation which can be achieved by scale and translation invariance property. By the way, he generalized the sub-additivity to an extended domain of U and V as $U, V \in \{kh, (k+1)h, \dots, (n+k)h\}$ where $h > 0$ and $k \in \mathbb{N}$.

It's clear that the proof is not completely analytical and generalization by approximation is not sufficient.

On the other hand, Wirch & Hardy (2000) suggested another theorem for sub-additivity of distorted risk measures by using concavity of distortion function.

1.1.14 Theorem (Wirch & Hardy, 2000)

The distortion risk measure $Hg(X) = \int_0^\infty g(SX(x))dx$ *is sub-additive if and only if,* g *is a concave distortion function.*

In the proof, authors brought some arguments in continuation of Wang (1995) and Wang (1996), which all are based on a theorem in Denneberg (1994) and is repeated in Kaas et al (2005). There have been chosen some amounts of g and then it has been proved that for a convex function of g there would be no place for sub-additivity for these chosen values. This completes the one way of theorem (if) and although, the referred terms are strictly true, but the proof is incomplete and vital point is that there is no proof for "only if" term!

By the way, it is necessary to bring a complete proof of sub-additivity property of distorted risk measures that is specialized here for distorted premium principle H_g.

We now turn to show that the premium principle H_g is subadditive. The proof is closely base on the theorems and propositions in Wang & Dhaene (1998) and (Dhaene, et al; 1999).

1.1.1 Lemma

Suppose that X *is a risk random variable and that* U *is a random variable which is uniformly distributed over the interval* $[0,1]$. *Then* X *and* $F_X^{-1}(U)$ *have the same distribution.*

Proof:

For this we must show that for every x,

$$\Pr(F_X^{-1}(U) \leq x) = \Pr(X \leq x)$$

but from the definition of the generalized inverse we have

$$\begin{aligned}\Pr(F_X^{-1}(U) \le x) &= \Pr(U \le F_X(x)) \\ &= F_X(x) \\ &= \Pr\,(X \le x)\end{aligned}$$

proving the claim.

■

1.1.2 Lemma

For every two risk variables X and Y and for U which is uniformly distributed over $[0,1]$, the variables $\{F_X^{-1}(U)\ ,\ F_Y^{-1}(U)\}$ are comonotone.

Proof:

In fact, the functions F_X^{-1} and F_Y^{-1} are increasing functions. Therefore, by a theorem above the random variables $F_X^{-1}(U)$ and $F_Y^{-1}(U)$ are comonotone.

■

Know consider the below proposition:

1.1.3 Proposition

For any two random variables X and Y and for the uniformly distributed $U \sim Uniform(0,1)$, we have the following inequality between the joint distributions

$$F_{X,Y}(x,y) \le F_{F_X^{-1}(U),F_Y^{-1}(U)}(x,y)$$

Proof:

$$\begin{aligned}F_{X,Y}(x,y) &= \Pr\,(X \le x, Y \le y) \\ &\le \begin{cases}\Pr\,(Y \le y) \\ \Pr\,(Y \le y)\end{cases}\end{aligned}$$

So,

$$F_{X,Y}(x,y) \le \min\{F_X(x), F_Y(y)\}$$

Now,

$$\begin{aligned} F_{F_X^{-1}(U),F_Y^{-1}(U)}(x,y) &= \Pr[F_X^{-1}(U) \le x,\ F_Y^{-1}(U) \le y] \\ &= \Pr\,[U \le F_X(x), U \le F_Y(y)] \\ &= \Pr\,[U \le \min\,\{F_X(x), F_Y(y)\}] \\ &= \min\{F_X(x), F_Y(y)\} \qquad (U \text{ is Uniformly distributed}) \\ &\ge F_{X,Y}(x,y) \end{aligned}$$

■

Remark: *In the literature, this fact is deviated by* ***Correlation Order*** *as,*

$$(X,Y) \le_{Corr} (F_X^{-1}(U), F_Y^{-1}(U))$$

4-1

1.1.4 Lemma

For $\forall d > 0$ *and any two random variables* X *and* Y *we have*

$$E[(X+Y-d)^+] = E(X) + E(Y) - d + \int_0^d F_{X,Y}(t, d-t)dt$$

Proof:

If $\begin{cases} 0 \le d - x - y, \\ 0 \le x, 0 \le y \end{cases}$, then $x \le d - y$. So then,

$$\int_0^d I_{[x,d-y]}(t)dt = \int_x^{d-y} dt = d - x - y = (d-x-y)^+.$$

But if $\begin{cases} 0 \le d - x - y, \\ 0 \le x, 0 \le y \end{cases}$, then the interval $[x, d-y]$ is empty, so $I_{[x,d-y]} = 0$. Thus,

$$\int_0^d I_{[x,d-y]}(t)dt = 0 = (d-x-y)^+.$$

Therefore, in either cases the inequality $\int_0^d I_{[x,d-y]}(t)dt = (d-x-y)^+$ holds. Thus if $\mathcal{P}$ is the probability measure on the sample space Ω, where X and Y act, then

$$\begin{aligned}
E[(X+Y-d)^+] &= \int_\Omega (d-x-y)^+ dP \\
&= \int_\Omega \int_0^d I_{[x,d-y]}(t)dtdP \\
&= \int_0^d \int_\Omega I_{[x,d-y]}(t)dP\,dt \qquad \text{Fubbini's Theorem} \\
&= \int_0^d \Pr[X \le t \le d-Y]dt \\
&= \int_0^d \Pr[X \le t, Y \le d-t]dt \\
&= \int_o^d F_{X,Y}(t, d-t)\,dt \qquad (*)
\end{aligned}$$

4-2

But we have,

$$\begin{aligned}
(X+Y-d)^+ &= X+Y-d+(X+Y-d)^- \\
&= X+Y-d+(d-X-Y)^+
\end{aligned}$$

so then by taking the expectation and using the result of equation (4-2), we can obtain

$$E[(X+Y-d)^+] = E(X) + E(Y) - d + \int_0^d F_{X,Y}(t, d-t)dt$$

■

1.1.5 Definition (Stop-loss order)

A risk X is said to precede a risk Y in stop-loss order, written $X \leq_{sl} Y$, if for all retentions $d \geq 0$, the net stop-loss premium for risk X is smaller than that for risk Y:

$$E[(X-d)^{+}] = E[(Y-d)^{+}]$$

1.1.6 Theorem

For every two random variable X and Y and for the uniformly distributed $U \sim Uniform(0,1)$, we have

$$X + Y \leq_{SL} F_X^{-1}(U) + F_Y^{-1}(U)$$

Proof:

We must show that for all d,

$$E[(X+Y-d)^{+}] = E[(F_X^{-1}(U) + F_Y^{-1}(U) - d)^{+}]$$

To show this, one first notes that from lemma (2.1.1) both random variables X and $F_X^{-1}(U)$ have the same distribution. Then, $E(X) = E[\,F_X^{-1}(U)]$ and $E(Y) = E[\,F_Y^{-1}(U)]$. Further, from proposition (2.1.3), we have

$$\begin{aligned} F_{X,Y}(t, d-t) &\leq F_{F_X^{-1}(U), F_Y^{-1}(U)}(t, d-t) \\ \int_0^d F_{X,Y}(t, d-t)\,dt &\leq \int_0^d F_{F_X^{-1}(U), F_Y^{-1}(U)}(t, d-t)\,dt \end{aligned}$$

Now the theorem follows from lemma (2.1.4).

■

1.1.7 Theorem (Sub-additivity)

For every two random variable X and Y and for the premium principle $H = H_g$, we have

$$H_g[X+Y] \leq H_g[X] + H_g[Y]$$

Proof:

Since H preserves the stop-loss ordering and since H is additive on comonotone risks, we can write

$$\begin{aligned} H[X+Y] &\leq H[F_X^{-1}(U)+F_Y^{-1}(U)] \\ &= H[F_X^{-1}(U)]+H[F_Y^{-1}(U)] \\ &= H[X]+H[Y] \qquad \text{as } X \text{ is distributed identically as } F_X^{-1}(U) \end{aligned}$$

■

1.1.8 Corollary

For any random variable U*, uniformly distributed on* $[0,1]$*, and any risks* $X_1, X_2, \ldots, X_n$*, we have*

$$\sum_{i=1}^{n} X_i \leq_{SL} \sum_{i=1}^{n} F_{X_i}^{-1}(U)$$

Note that $X_1, X_2, \ldots, X_n$ and $F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)$ have the same marginal distribution, while the risks $F_{X_i}^{-1}(U)$, $i = 1, 2, \ldots, n$, are mutually comonotone generalized lemma (2.1.2) for n random variables. This corollary states that in the class of all multivariate risks $(X_1, X_2, \ldots, X_n)$ with given marginals, the stop-loss premiums of $(X_1 + X_2 + \ldots + X_n)$ are maximal if the risks X_i are mutually comonotone.

More generally, one can obtain sub-additivity for more than two random variables. The multivariate case follows immediately by considering the fact that $H[X+Y+Z]$ can be sub-additive with respect to $H[X+Y]$ and $H[Z]$. In the same way, $H[X_1 + X_2 + \ldots + X_{n-1} + X_n]$ is sub-additive respecting $H[X_1 + X_2 + \ldots + X_{n-1}]$ and $H[X_n]$. Thus,

$$H[X_1 + X_2 + \ldots + X_{n-1} + X_n] \leq \sum_{i=1}^{n} H[X_i]$$

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